"Hot X: Algebra Exposed"

Supplemental PDF: <u>Proof of Quadratic Formula by completing the square</u>

...as promised on p. 396

So, in Chapter 25 of *Hot X: Algebra Exposed*, I showed you how to "complete the square" on any quadratic equation. Yep, if your equation looks like this: $ax^2 + bx + c = 0$, where a, b, c are integers, then you can complete the square on it, and rewrite it into the form: $(x + d)^2 = e$, where d and e are also integers. This makes it much easier to solve for x, (assuming that real solutions exist!) since we can then take the square root of both sides and isolate x, which is always a good thing when you're trying to solve for it. ☺

(Do yourself a big favor and re-read Chapter 25 until you feel comfortable with these techniques before trying to get through this PDF. You'll thank me later.)

Then in Chapter 26, I showed you how to use the quadratic formula, which also can solve any quadratic equation that has real solutions, and I showed you how to easily memorize it by thinking about Betty at the mall and all her adventures there with the snobby "popular" girls, remember?

But where did this big thing *come* from?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As it turns out, we can derive this formula by completing the square on the generic equation $\boldsymbol{ax^2 + bx + c = 0}$. Let's do it! We're going to use the "completing the square" steps outlined on p.374 in *Hot X: Algebra Exposed*, so I highly recommend opening to that page and following along. By the way, first you should totally practice the completing the square technique on equations with actual numbers; this proof would be pretty darn advanced if you weren't already familiar with the technique!

Ok, so according to the steps, the first thing we'll do to this equation is to multiply both sides by $\frac{1}{a}$, using the distributive property:

$$\frac{1}{a}\left(ax^2 + bx + c\right) = \frac{1}{a}(0)$$

$$\rightarrow\ x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Don't let the fractions get you down! Then we subtract $\frac{c}{a}$ from both sides, so that only the stuff that's multiplied by x is on the left, and only *constants* are on the right:

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

So far so good? Great. Next, we write blank spots where the new "magic" squaure number will go on both sides:

$$x^2+\frac{b}{a}x\ +\ \_\_\_\_\ =\ -\frac{c}{a}+\ \_\_\_\_$$

What number should we add to both sides? We take x's cofficient (which is $\frac{b}{a}$), divide it by 2 (we make a mental note that this is $\boldsymbol{\frac{b}{2a}}$, so that's what will end up going in the <u>parentheses</u>) and then square it. So our magic missing number is $\frac{b^2}{4a^2}$, right? And now let's add that to both sides of the equation, which of course doesn't change the trueness of the equation at all; it just makes it even messier-looking:

$$x^2+\frac{b}{a}x\ +\ \frac{b^2}{4a^2}\ =\ -\frac{c}{a}+\frac{b^2}{4a^2}$$

Yes, things are getting crazier, but stick with me if you can!

What's next? Well, now we've actually completed the square; we've written the left side into something that <u>can be</u> written as a squared parentheses, so let's do it. As we've practiced, this parentheses will be filled with x and the magic number *before* it was squared (the bolded one above):

$$\left(x+\frac{b}{2a}\right)^2\ =\ -\frac{c}{a}+\frac{b^2}{4a^2}$$

If you want, just multiply out the left side, and you'll see we have the same exact equation as above it. Why did we do all this? So that we can isolate x, which we now can more easily do, by just taking the square root of both sides of this equation and getting rid of that pesky exponent that is keeping the x trapped inside the parentheses!

But first, let's make things nicer-looking by combining those two fractions on the *right* side of the equation, $-\frac{c}{a}+\frac{b^2}{4a^2}$, using techniques that I showed you in Chapter 5 of *Hot X: Algebra Exposed*. So, to use a common denominator of $4a^2$, we'll multiply $-\frac{c}{a}$

by the copycat $\frac{4a}{4a}$ and add $\frac{-4ac}{4a^2}+\frac{b^2}{4a^2}$ to get $\frac{-4ac+b^2}{4a^2}$ for the right side of our big equation. We can reverse the order of the two terms on top of the fraction to get: $\frac{b^2-4ac}{4a^2}$, and now our entire, big equation looks like this:

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

With me so far? Yeah, you're a total rock star. By the way, if you've been working with the quadratic formula for awhile, then things should start to be looking familiar…

Ok, now that we've simplified the right side of the equation, we're ready to take the square root of both sides and isolate *x*. Let's see what happens:

$$\sqrt{\left(x+\frac{b}{2a}\right)^2} = \sqrt{\frac{b^2-4ac}{4a^2}}$$

Yeah, um, this is getting ridiculous, but let's keep going anyway!

For a moment, let's notice that we are taking the square root of an equation that involves a squared *variable* on one side, which means we must stick $\pm$ on the other side. We talked all about this in Chapter 25 of *Hot X: Algebra Exposed*.

So, just like on p.370, where I showed you how:

$$\sqrt{x^2} = 25$$

$$\rightarrow x = \pm 5$$

and on p.371, where I showed you that:

$$\sqrt{(w-2)^2} = 3$$

$$\rightarrow w-2 = \pm\sqrt{3}$$

(and check out p.381-382 to see *why* that sort of thing is true)

…in this case, we apply the same thinking, and since there is a variable on the left, when we take the square root of both sides (and the parentheses' exponent and square root "cancel" each other on the left), we need to stick $\pm$ on the *right* side:

$$\sqrt{\left(x+\frac{b}{2a}\right)^2} = \sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\rightarrow \quad x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

(Pant, pant!)

This is JUST like the "*w*" example above it. Remember, all that *a*, *b*, *c* stuff is just numbers! Now, the only thing left to do in order to isolate *x* is to subtract $\frac{b}{2a}$ from both sides and simplify:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2-4ac}{4a^2}}$$

This looks so close to the quadratic formula, but it's not quite there yet. Why? Because we can simplify inside that square root symbol! See, the denominator, $4a^2$, is a perfect square, so we can rewrite this as (see Chapter 19 to review simplifying radical expressions):

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

And hey, now we have a common denominator and can combine these two fractions:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

And voila! We've used the "completing the square" technique to actually *derive* the quadratic formula.

Ok, this was WAY advanced, and I'm so incredibly proud of you for reading it, I don't even know what to say. Don't worry if there were parts you stumbled on, by the way – it's a lot to get through!

But if you want to strengthen your brain (and totally impress your teacher and me), then read this proof a few more times until you really understand it. Next step? Print it out and explain it to someone else.

And then? Try it on your own, without looking! When you are able to do that, I want you to email me. I'll be so proud!